PERGAMON

International Journal of Solids and Structures 38 (2001) 6997–7013

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Strain energy density failure criterion

Q.M. Li *

Division of Structures and Mechanics, School of Civil and Structural Engineering, Nanyang Technological University, Nanyang Avenue, Singapore 639798, Singapore

Received 18 October 1999

Abstract

The initiation of macroscopic material failure is associated with the collective disruption of atomic bonds, which is driven by the potential energy stored in the atomic bonds. This potential energy, which is represented by specified elastic strain energy density in a mechanical system, is releasable. Thus, a universal macroscopic material failure criterion in a mechanical system can be defined by a specified elastic strain energy density together with its critical value that is determined by preceding irreversible deformation process and current environmental state. A dissipative function based on continuum mechanics and irreversible thermodynamics is proposed to represent the irreversible deformation process. The increase of this dissipative function due to material inelastic deformation, damage and other possible intrinsic dissipative mechanisms in a mechanical system leads to the reduction of material strength. When the material failure is dominated by the dissipation, a dissipative energy density failure criterion can be defined by using the dissipative function. On the other hand when the intrinsic dissipation is negligible during the deformation process before failure, the specified elastic strain energy density and its critical value, which is determined by the initial material bond strength, can be used to define material brittle failure. It also shows the possibility to set up a relationship between fracture mechanics and failure criterion. The proposed method to represent failure criteria is based on continuum mechanics and irreversible thermodynamics and retrieves previously successful failure criteria.

Keywords: Strain energy density; Failure criterion; Continuum mechanics; Dissipation

1. Introduction

Many failure criteria have been proposed and used successfully to predict the initiation of macroscopic material failure under various loading conditions. The physical foundation for these failure criteria might be different, and their individual validity may be restricted to specified types of material and structural failure problems. Generally speaking, macroscopic failure criteria may be classified into four different types, (a) stress or strain failure criteria; (b) energy type failure criteria; (c) damage failure criteria; and (d) empirical failure criteria. Among these failure criteria, the plastic strain energy density failure criterion has

* Fax: +65-791-0676.

E-mail address: cqmli@ntu.edu.sg (Q.M. Li).

PII: S0020-7683(01)00005-1

been recalled recently to predict failure initiation in various structural elements when subjected to dynamic loads, which was summarised by Jones and Shen (1993) and Jones (1997). The purpose of the present paper is to discuss the theoretical foundation of a strain energy density failure criterion and its relationships with other failure criteria.

Using a potential energy concept to define material failure was proposed initially by Huber (see Freudenthal (1950, p. 389)). Hill (1948) established extreme principles of material plasticity by using energy concept. It is Freudenthal (1950) who clearly realised that material failure in different material scales occurs in different forms simultaneously and only the energy concept is universal throughout all the material scales involved. This realisation led him to introduce an energy concept to describe material failure. The importance of the energy concept was also emphasised by Gordon (1976).

Furthermore, Freudenthal noticed the difference between elastic deformations (related to potential energy) and inelastic deformations (dissipated energy) in an energy type failure criterion, which is expressed by the following statement (Freudenthal (1950, p. 20)):

"Since the process of bond separation which initiates fracture depends on the momentary elastic strain or the potential energy in a different manner from what it depends on the inelastic strain or the dissipated energy, the elastic and inelastic strain energy must enter the fracture condition separately".

While, Gordon (1976, p. 71) also pointed out that

"The weakening mechanism, rather than the bond strength, is what really controls mechanical strength".

Although several types of strain energy failure criteria have been proposed after Freudenthal, only few of them, for example, Lemaitre and Chaboche (1990, pp. 402–403), expressed similar viewpoint. Thus, it is necessary to emphasise the fundamental difference between the contributions of elastic strain energy and inelastic strain energy in an energy failure criterion.

The aim of the present paper is to discuss the energy type failure criteria based on continuum mechanics and irreversible thermodynamics. It is not the purpose of the present paper to propose any new failure criteria and to verify their practical applications, which have various forms in different application fields. This and other two published works, Li (1999) and Li (2000), belong to a systematic study on energy roles in material deformation and failure.

The specified elastic strain energy density failure criterion is discussed in Section 2 from Freudenthal (1950)'s original work. The influence of an irreversible process, described by a dissipative function based on thermodynamics, on the critical state of material cohesive potential, is also studied in this section. This function is used to define a dissipative energy density failure criterion when material failure is controlled by inelastic dissipation, which reduces to several existing successful ductile failure criteria, as discussed in Section 3. An expression of the inelastic strain energy density failure criterion in two-dimensional structural elements is discussed in Section 4 followed by discussion on relationship between fracture mechanics and failure criterion in Section 5.

2. Strain energy density failure criterion

2.1. Potential and dissipative energies in a mechanical system

The material deformation and failure in reaction to applied force are associated with deformation and damage mechanisms from atomic to macroscopic levels, as shown in Fig. 1. The input mechanical energy will be either stored or consumed in a solid continuum through different deformation and failure mechanisms. Material deformation and failure at different levels have different physical meanings, which are correlated between different levels by proper statistical methods. Generally speaking, the material behaviour on one level is a collective, or group, behaviour of the material on a sub-level. A successful deformation and failure theory should be physically consistent at each material level.

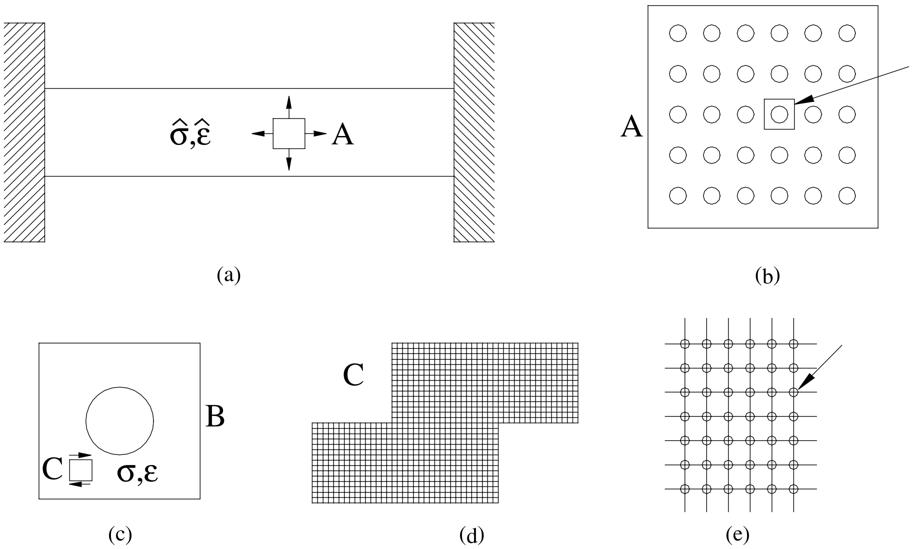

Fig. 1. (a) Structural element scale; (b) macroscopic scale, where macroscopic stress and strain are defined; (c) mesoscale, which is represented by a typical void; (d) microscale; (e) atomic scale.

Macroscopic material deformations may be divided into elastic and inelastic deformations corresponding to different mechanical mechanisms at the microscopical level. The free-electron theory of metals determines the metal behaviour at atomic level, which suggests that the valence electrons in a metallic crystal are detached from their atoms and can move freely between the positive metallic lattice and the negative free electrons to provide the cohesive strength of the metal. Such a linkage may be regarded as a special case of covalent bonding through a valence electron cloud. The equilibrium space between two bonded ions is determined by a balance between the attraction due to the bond and the repulsion that develops when their outer electron shells begin to overlap each other. Thus, atoms in condensed phase occupy equilibrium positions at the valley of the inter-atomic energy curve (Fig. 2) determined by the formed atomic bonds (Woo and Li, 1993). Upon the application of external force, the atoms are displaced from their equilibrium positions, which changes the potential energy of the system and is recoverable with the removal of external force within a certain range of displacement. Macroscopic elastic deformation is a collective behaviour of atomic displacements from their equilibrium positions. And, the macroscopic stress, defined as the material interactions on the level of continuum mechanics, is the collective behaviour of interactions between atomic bonds.

On the other hand, atoms may overcome their energy barriers and move into a new equilibrium valley of free energy with excessive input of external mechanical energies, which leads to the breaking of the previous bonds and re-establishing a new configuration of bonds. Material defects, or material damages, will be formed during this re-bonding process. A collective atomic migration in a stress field causes a slip (or dislocation) in a crystal, whose collective behaviour leads to macroscopic plastic (or inelastic) deformations. Furthermore, the atomic bond may be disrupted when the piled up dislocation barriers during atomic migration process obstruct atomic migrations. Therefore, macroscopic inelastic deformation is an important mechanism to weaken the macroscopic material cohesive potential.

It is evident that failure on the atomic level does not imply a material failure on macroscopic level. However, both the atomic migration and atomic bond disruption cause a change of the material structures, and, therefore, a change of the material macroscopic property and strength. For ductile materials, void

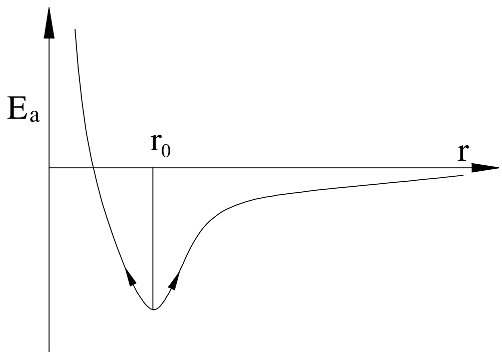

Fig. 2. The inter-atomic energy curve, r is the distance between atoms.

developments, which represent a material structural change from the macroscopic level viewpoint, are significant under certain conditions. The common feature of inelastic deformation and material damage is their irreversibility, which is a dissipative process from thermodynamics viewpoint. Thus, the macroscopic failure process of a material should be described by the thermodynamic theory of irreversible processes.

2.2. Thermodynamic foundation of dissipative process

Material failure is a process involving thermal and mechanical phenomena from atomic to macroscopic levels. The thermodynamics of a continuum medium determines the thermal and mechanical behaviours of solid continuum in its deformation and failure process. It has been noted in Section 2.1 that any changes of material structures related to macroscopic failure process might be described by irreversible thermodynamic theory.

The Clausius–Duhem inequality is

$$\sigma_{ij}\dot{\varepsilon}_{ij} - \rho(\dot{\Psi} + s\dot{T}) - q_i\frac{T_{,i}}{T} \geqslant 0 \tag{1}$$

when the specific free energy $\Psi = e - Ts$ is introduced, in which σ_{ij} is the Cauchy (true) stress, $\dot{\varepsilon}_{ij}$ is the rate of deformation, ρ is the material density, e is the specific internal energy of the material, s is the specific entropy, T is temperature, and $\vec{q}$ is the heat flux vector. The specific free energy Ψ is determined completely by σ_{ij} (or ε^{e}_{ij}) and other state variables, i.e.,

$$\Psi = \Psi(\sigma_{ij},\ \text{other state variables}) \quad \text{or} \quad \Psi = \Psi(\varepsilon^{e}_{ij},\ \text{other state variables}), \tag{2a, b}$$

in which strain is determined uniquely by deformation gradient. In Eq. (2a,b), the so-called "other state variables" may be represented by a group of observable and internal variables, T, $V_0 = D_{ij}$ (as damage internal variables), V_k $(k = 1, 2, \ldots)$ (as plastic internal variables). Thus, we have

$$\Psi = \Psi(\varepsilon_{ij}, \varepsilon^{p}_{ij}, T, D_{ij}, V_k) \tag{3}$$

because ε^{e}_{ij} can be expressed as a function of ε_{ij} and ε^{p}_{ij} according to the multiplicative decomposition (Lee and Liu, 1967; Lee, 1969). Eq. (3) gives

$$\dot{\Psi} = \frac{\partial\Psi}{\partial\varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial\Psi}{\partial\varepsilon^{p}_{ij}}\dot{\varepsilon}^{p}_{ij} + \frac{\partial\Psi}{\partial T}\dot{T} + \frac{\partial\Psi}{\partial D_{ij}}\dot{D}_{ij} + \frac{\partial\Psi}{\partial V_k}\dot{V}_k. \tag{4}$$

Thus, the Clausius–Duhem inequality, Eq. (1), predicts

$$\sigma_{ij} = \rho \frac{\partial \Psi}{\partial \varepsilon_{ij}}, \quad s = -\frac{\partial \Psi}{\partial T}, \quad P_{ij}\dot{\varepsilon}^{\mathrm{p}}_{ij} + Y_{ij}\dot{D}_{ij} + A_k \dot{V}_k - \frac{1}{T} q_i T_{,i} \geqslant 0, \tag{5a–c}$$

where $P_{ij} = -\rho\,\partial\Psi/\partial\varepsilon^{\mathrm{p}}_{ij}$, $Y_{ij} = -\rho\,\partial\Psi/\partial D_{ij}$ and $A_k = -\rho\,\partial\Psi/\partial V_k$ are thermodynamic forces (stresses) corresponding to intrinsic dissipative processes represented by $\varepsilon^{\mathrm{p}}_{ij}$, D_{ij} and V_k. σ_{ij}, determined from Eq. (5a), has a one-to-one relationship with ε_{ij} for given T, $\varepsilon^{\mathrm{p}}_{ij}$, D and V_k within a non-discontinuous stress set. It can be proved that

$$\frac{\partial \Psi}{\partial \varepsilon_{ij}} = -\frac{\partial \Psi}{\partial \varepsilon^{\mathrm{p}}_{ij}}, \tag{6}$$

and therefore, $P_{ij} = \sigma_{ij}$, which means that the macroscopic Cauchy stress is the thermodynamic force of plastic dissipation when it is represented by plastic strain (Li, 1999).

The dissipation in a mechanical system can be described by Eq. (5c) according to the thermodynamic principles. When intrinsic mechanical dissipation and thermal dissipation due to heat conduction are decoupled according to Truesdell and Noll (Malvern (1969, p. 256)), the intrinsic mechanical dissipation rate is expressed by

$$\dot{\phi}_1 = P_{ij}\dot{\varepsilon}^{\mathrm{p}}_{ij} + A_k \dot{V}_k + Y_{ij}\dot{D}_{ij} \geqslant 0. \tag{7}$$

Function ϕ_1 in Eq. (7) describes an intrinsic dissipative process in a mechanical system (Li, 1999), which will be used as an intrinsic scale to measure material's life expectancy before macroscopic failure.

In a mechanical system, P_{ij}, Y_{ij} and A_k are defined as a group of thermodynamic stresses. Each thermodynamic stress corresponds to an internal variable that introduces a mechanical intrinsic dissipative mechanism. Their contributions to the total mechanical intrinsic dissipation are reflected in Eq. (7).

Now, the mechanical dissipative energy density, ϕ_1, is defined by

$$\phi_1 = \int_0^t \dot{\phi}_1 \, \mathrm{d}t, \tag{8}$$

which may be used as an index to describe the decrease of material bonding strength at the macroscopic level within the scope of a thermodynamic framework.

2.3. Specified elastic strain energy density failure criterion

According to Freudenthal (1950) and the above discussion the macroscopic material failure criterion may be described by the following statement:

> *When the specified potential strain energy* [1] *over a given collection of particles of the material* [2] *exceeds a critical value, which depends on the preceding irreversible dissipative process and some environmental parameters, the macroscopic material failure occurs in this collection of material particles.*

It is evident that a proper statistical average process is necessary to express this failure criterion within the scope of continuum mechanics. It has been shown that, despite the large difference in the nature and structure of materials, there is a considerable unity displayed in their macroscopic behaviour, described by

[1] The specified potential strain energy may be understood as the specified elastic strain energy which refers to the maximum possible releasable strain energy in the given material collection.

[2] A given collection of material particles can be identified using material (or Lagrangian) descriptions according to initial material configuration. The choice of the size of a given collection of material particles depends on the described physical properties and material constructions.

macroscopic physical quantities, such as material density, material elasticity, etc., which are defined in average over a representative volume of material. Together with the limitation concept in mathematics this technique defines all meaningful physical quantities in continuum mechanics at a geometrical point. However, the geometrical point in continuum mechanics, on which macroscopic physical quantities are defined, should be sufficiently large on the microscopic level. Hancock and Mackenzie (1976) noted that failure initiation must involve a minimum amount of materials based on the characteristic scale of the physical events. Two important issues might arise from using this average technique, i.e., (a) scaling and size effects on determining material failure index (Bazant and Chen, 1997), and (b) FE size effect on applying failure criterion to predict failure initiation. These two issues are associated not only with the currently studied energy failure criterion, but also with all other macroscopic failure criteria. Here, we assume that such an average technique is valid for describing macroscopic material failure in continuum mechanics. Therefore, the proposed failure criterion may be described by

$$w_s^e = w_{sc}^e(\phi_1, \dot{\phi}_1, T), \tag{9}$$

where w_s^e is the specified elastic strain energy density defined on the original representative volume [3], w_{sc}^e is its critical value, ϕ_1 and $\dot{\phi}_1$ are the mechanical dissipative energy density defined on the original representative volume of the material and its rate, respectively, T is the temperature. It should be noted that if Cauchy stress and natural strain are used, the volume change should be considered in calculating w_s^e. Similar attentions should be paid to calculate ϕ_1 and $\dot{\phi}_1$.

2.4. Material brittle failure

It is well known that material under low temperature, high stress triaxiality or high strain rate tends to fail in a brittle manner with only a small amount of prior plastic flow. In this case, the influence of internal mechanical dissipation, ϕ_1, on w_{sc}^e in Eq. (9) may be neglected. Furthermore, existing investigations have shown that the mechanism of brittle failure, i.e., cleavage, is a mechanism involving the breakage of inter-atomic bonds. This mode of failure is not very sensitive to temperature and strain rate according to the Ludwik–Davidenkov–Orowan hypothesis as discussed by Cottrell (1964), Dodd and Bai (1987, pp. 51–71) and Polakowski and Ripling (1966, p. 230). Therefore, the failure criterion, Eq. (9), can be simplified into

$$w_s^e = w_{sc}^e \tag{10}$$

for brittle failures. Within the scope of linear elastic mechanics, the influence of volume change on the specified elastic strain energy density may be neglected because the relative volume change is

$$J^e = \frac{3\sigma_H}{E}(1-2v) \sim \frac{\sigma_H^c}{E} \sim 10^{-2} \tag{11}$$

for most metals, where σ_H^c is the critical value of hydrostatic tensile stress for material failure.

The different choice of a specified elastic strain energy density leads to different types of stress failure criteria which have been used widely to calculate the material strengths in brittle failure analyses.

(1) *Maximum normal stress failure criterion:* Material failure is initiated by the potential of strain energy density related to the specified normal stress that is responsible for initiating failure

$$w_s^e = \frac{\sigma_1^2}{E} - \frac{v}{E}(\sigma_1\sigma_2 + \sigma_1\sigma_3) = \frac{1+v}{E}\sigma_1^2 - \frac{3v\sigma_H}{E}\sigma_1, \tag{12}$$

[3] Material mass is more proper to represent the given collection of material particles (Matic et al., 1988), which is equivalent to using their original geometrical volume due to the conservation of mass.

where σ_1 is the principal stress to control material failure. Therefore, Eqs. (10) and (12) are equivalent to

$$\sigma_{\max} = \sigma_{1\mathrm{f}} \tag{13}$$

which is the maximum normal stress failure criterion.

(2) *von-Mises failure criterion:* When material failure is initiated by the potential of distortion strain energy density as proposed by Freudenthal (1950, pp. 387–394)

$$w_{\mathrm{s}}^{\mathrm{e}} = w_{\mathrm{d}}^{\mathrm{e}} = \frac{4(1+v)}{6E}\sigma_{\mathrm{e}}^2, \tag{14}$$

Eq. (10) may be expressed by

$$\sigma_{\mathrm{e}} = \sigma_{\mathrm{ef}} \tag{15}$$

which is the von-Mises or octahedral shearing stress failure criterion. It should be noted that Eq. (15) has also been used as plastic yielding criterion for metals.

(3) *Maximum shear stress failure criterion:* The elastic strain energy density associated with the maximum shear stress is

$$w_{\mathrm{s}}^{\mathrm{e}} = \int_0^{\gamma} \tau_{\max}\mathrm{d}\gamma = \frac{\tau_{\max}^2}{2G} = \frac{(\sigma_1-\sigma_3)^2}{8G}, \tag{16}$$

with G as shear modulus, so that, Eq. (10) reduces to the maximum shear stress failure criterion, i.e.,

$$\tau_{\max} = \frac{\sigma_1-\sigma_3}{2} = \tau_{\mathrm{f}}. \tag{17}$$

(4) *Hydrostatic tension failure criterion:* In this case, the corresponding elastic strain energy density is

$$w_{\mathrm{s}}^{\mathrm{e}} = w_{\mathrm{v}}^{\mathrm{e}} = \frac{3(1-2v)}{2E}\sigma_{\mathrm{H}}^2, \tag{18}$$

therefore, Eq. (10) is equivalent to

$$\sigma_{\mathrm{H}} = \sigma_{\mathrm{H}}^{\mathrm{c}}, \tag{19}$$

where $\sigma_{\mathrm{H}}^{\mathrm{c}}$ is the critical value of the hydrostatic tension stress.

These failure criteria are commonly used to estimate the failure strength of various materials, and may be generalised as a stress type failure criterion

$$h(\sigma_{ij}) = h_{\mathrm{c}} \tag{20}$$

which could be simplified further by using material isotropic properties.

This type of failure criterion is suitable for the failure of brittle material due to its independence of the deformation history. The key issue is to define the specified elastic strain energy density that controls a particular type of brittle failure. More discussion on the stress type failure criteria was presented in Polakowski and Ripling (1966), Dorn (1948) and Nadai (1950, pp. 207–228). Eq. (20) may also be used in predicting ductile material failures when the influence of deformation history is retained in the critical value on the right side of a failure criterion equation, as discussed by Dorn (1948). Dorn's conclusion is consistent with the present expression in Eq. (9) when the deformation history is understood as an irreversible dissipative process. In a plane stress state, these failure criteria are frequently described by biaxial failure maps in metal-forming studies. It should be noted that using a stress type failure criterion to predict a general ductile material failure is difficult because of the importance of the strain history effects in ductile failure processes. However, when the loading path or deformation pattern is certain, a stress type failure criterion is useful for predicting ductile material failures. Thus, biaxial failure maps have been accepted in the metal-forming industry where the loading path is known for each working process. The role of stress and strain state in a ductile failure was discussed by Hancock and Brown (1983).

3. Dissipative energy density failure criterion associated with large inelastic deformation and material failure

3.1. Dissipative energy density failure criterion

$w^{\rm e}_{\rm sc}$, used to represent material strength in Eq. (10), decreases during an irreversible dissipative process, i.e., $\partial w^{\rm e}_{\rm sc}/\partial\phi_1 < 0$. Therefore, material ductile failure is a progressive process represented by the continuous decrease of the material strength with the continuation of the dissipation. Eventually, material ductile failure emerges as a natural outcome of excessive internal dissipation in a mechanical system. After a sufficiently large dissipation that normally corresponds to large inelastic deformations, material failure is controlled mainly by the material dissipations represented by ϕ_1 instead of being determined by internal interactions among atoms in the case of small inelastic deformations although these internal interactions are still the active driving force to separate the material. Experimental studies, for example Bridgman (1952), have shown that the hydrostatic stress has a strong influence on material failure while its influence on plastic deformation is negligible. Tensile triaxiality decreases the ductility of a material, while a compressive triaxiality increases it. These effects should be reflected on the left side of Eq. (9). Thus, the hydrostatic stress should be considered as an environmental parameter that may influence the critical value of ϕ_1 when other elastic actions are neglected. If Ludwik–Davidenkov–Orowan hypothesis (see Section 2.4) is applicable for inelastic deformation process, the failure criterion for ductile material failure in large inelastic deformations becomes

$$\phi_1 = \phi_{1\rm c}(\sigma_{\rm H}) \tag{21}$$

according to Eq. (9), where, $\phi_{1\rm c}$ is the critical value of the mechanical dissipative energy density. Eq. (21) is the dissipative energy density failure criterion, which may be used to predict material ductile failure under large inelastic deformations.

According to Eq. (7), the dissipation energy density is

$$\phi_1 = w^{\rm p} + w^{\rm d} - w^{\rm v}, \tag{22}$$

where, ϕ_1 is the intrinsic dissipative energy density, $w^{\rm p} = \int \sigma_{ij}\,{\rm d}\varepsilon^{\rm p}_{ij}$ is the inelastic strain energy density, $w^{\rm d} = \int B\,{\rm d}D$ is the damage dissipation density (from now on, Y_{ij} and D_{ij} in Eq. (7) are substituted by B and D when the isotropic damage is assumed) and $w^{\rm v} = \int(-A_k)\,{\rm d}V_k$ is the stored energy density of cold work (Chrysochoos and Belmahjoub, 1992). For metals, $w^{\rm v}$ is the energy of the field of the residual microstresses which accompany the increase in the dislocation density. It represents only 5–10% of the term $w^{\rm p}$ and is often negligible (Lemaitre and Chaboche 1990, p. 68). Therefore, Eq. (22) is simplified into

$$\phi_1 = w^{\rm p} + w^{\rm d}. \tag{23}$$

The possibility to use total dissipated energy to define a failure criterion was also proposed by Benallal et al. (1992) without further discussion.

If it is assumed that material plastic flow and material damage are two different physical processes, as shown by Lemaitre and Chaboche (1990), we have

$$\sigma_{ij}\dot{\varepsilon}^{\rm p}_{ij} \geqslant 0 \quad \text{and} \quad B\dot{D} \geqslant 0. \tag{24a, b}$$

Eq. (24a,b) implicates that the intrinsic dissipation consists of two different types, i.e., plastic dissipation and damage dissipation represented by two monotonously increasing functions, $w^{\rm p}$ and $w^{\rm d}$, and the effects of these two different processes on material failure may be different.

3.2. Plastic strain energy density failure criterion

Plastic strain energy density failure criterion has been used widely in predicting material ductile failures for various purposes. Cockcroft and Latham (1968) suggested that the critical value of the inelastic strain energy density at fracture is practically constant at moderate strain rates, which was used to study the fracture of solid polymers by Vinh and Khalil (1984). Gillemot (1976) used it to predict the crack initiation in materials. Clift et al. (1987, 1990), employed the finite-element technique and several failure criteria to predict the fracture initiation in a range of simple metal-forming operations. It turns out that only the plastic strain energy density failure criterion predicts successfully the correct fracture initiation sites, which were observed in the experimental results. A plastic strain energy density failure criterion was combined with dynamic rigid, perfectly plastic analyses of beams and circular plates by Shen and Jones (1992, 1993a,b) and Jones and Shen (1993), and good agreement was found with the actual dynamic failure in several structural elements. These investigations have shown the validity of this failure criterion in some circumstances. However, many other failure criteria like the damage failure criterion also give encouraging failure predictions. It appears that both the damage failure criterion and the plastic strain energy density failure criterion work well in many circumstances. But, it is not clear if they can be derived from same physical foundation.

Based on Eq. (24a,b), it is concluded that material ductile failure is controlled by two dissipative mechanisms, which are plastic deformation dissipation and damage dissipation, respectively. Material failure may occur due to either excessive plastic dissipation with small material damage from void development or excessive material damage from void coalescence. It is also possible that two dissipative mechanisms are important for the initiation of failure. If material failure is mainly controlled by the plastic deformation mechanism, the failure criterion will become a plastic strain energy density failure criterion where the contributions from voids are negligible for material failure. In this case, material incompressibility is approximately valid, i.e., $d\varepsilon_{11}^p + d\varepsilon_{22}^p + d\varepsilon_{33}^p = 0$, and therefore,

$$w^p = \int_0^{\varepsilon_{ij}^p} \sigma_{ij}\, d\varepsilon_{ij}^p = \int_0^{\varepsilon_{ij}^p} s_{ij}\, d\varepsilon_{ij}^p + \int_0^{\varepsilon_{ij}^p} \sigma_H (d\varepsilon_{11}^p + d\varepsilon_{22}^p + d\varepsilon_{33}^p) = \int_0^{\varepsilon_{eq}^p} \sigma_{eq}\, d\varepsilon_{eq}^p = w_c^p, \tag{25}$$

where $s_{ij} = \sigma_{ij} - \sigma_H \delta_{ij}$, w_c^p is the critical value of plastic strain energy density determined by experiments. Eq. (25) was termed as a generalised plastic work per unit volume in Clift et al. (1987, 1990), and is called a plastic strain energy density failure criterion in Jones and Shen (1993). Basically, plastic strain energy density failure criterion is valid for failure initiation controlled mainly by plastic dissipative process.

3.3. Continuum damage failure criteria

In this section, we will discuss the situation when the failure process is controlled mainly by material damage dissipation. The microscopic mechanism is the coalescence of voids due to the initiation of unstable flow between the neighbouring voids. In this case, the failure criterion is

$$w^d = w_c^d \quad \text{or} \quad w^e\left(\frac{\overline{E}}{E} - 1\right) = w_c^d, \tag{26}$$

where $w^d = w^e(\overline{E}/E - 1)$ according to damage mechanics, $\overline{E}$ is Young's modulus of the virgin material, w_c^d is the critical value of the damage dissipation density. The elastic strain energy, w^e, depends on strain hardening (also depends on strain rate hardening if it is significant), hydrostatic stress, σ_H and current Young's modulus, E, while, the current Young's modulus depends on the damage parameter, D. Thus, Eq. (26) may be expressed by

$$D = D_c(\sigma_H) \tag{27}$$

when material plastic strain hardening is not significant during a damage control failure process. Eq. (27) is well known as damage failure criterion and has been used widely in continuum damage mechanics to predict the initiation of material ductile failure. The significance of the parameter σ_H on the value of D_c needs to be verified by experiments. Alves (1996) examined the influence of σ_H on the critical damage parameter, D_c. It was concluded that the critical damage parameter does not seem to be highly stress-state dependent for the mild steel. A similar conclusion was reached by Otsuka et al. (1987) for a structural steel SM41A. However, other evidence gave controversial results (Becker, 1987; Shi et al., 1991). A discussion on this topic is given by Alves (1996). Strain rate and temperature influences on the critical value of D are not included in Eq. (27). It was also shown by Alves (1996) that strain rate does not significantly influence the critical damage parameter in the studied problem. However, it is well known that material ductility decreases with the increase of strain rate and the decrease of temperature. For a damaged material, material ductility includes both plastic deformation and damage development. Thus, it is highly possible that strain rate and temperature have some influences on the critical damage parameter, which need to be verified by further experiments.

Another potential theory, used to predict material failure, is the porous ductile material model established by Gurson (1977) and developed successively by many other authors. Relationships between porous ductile material model and continuum damage mechanics have been studied in Li (2000). In spite of the different modelling procedures used in these two theories, similar concepts are used in both theories to define material failure. In continuum damage mechanics, the surface density of the discontinuity of the material (D) is used to represent material damage and to define material failure by its critical value (D_c). In porous ductile material model, the void volume fraction (f) together with its critical value (f_F) is used to define the material failure. For an isotropic material, there is a relationship between them for a given void type (Li, 2000). Therefore, Eq. (27) can be expressed equivalently by

$$f = f_c(\sigma_H) \tag{28}$$

as used in many void growth models (Gurson, 1977; Tvergaard, 1990).

The application of Eq. (27) or Eq. (28) requires the solution of the evolution equations of a damage index, which depends on the development of macroscopic stress and strain and their histories, and, in turn, the macroscopic stress and strain contain the information of both plastic dissipation and damage dissipation. Therefore, it is possible to use macroscopic stress and strain to define a failure criterion, which have been proved by existing results.

3.4. Further discussion on dissipative energy density failure criterion

After defining a dissipative surface, $F = 0$, which distinguishes the elastic domain from other dissipative responses (Li, 1999), the thermodynamic principle and the minimum specific free energy principle lead to

$$\mathrm{d}\varepsilon_{ij}^{\mathrm{p}} = \mathrm{d}\lambda \frac{\partial F}{\partial \sigma_{ij}}, \quad \mathrm{d}D_{ij} = \mathrm{d}\lambda \frac{\partial F}{\partial Y_{ij}} \quad \text{and} \quad \mathrm{d}V_k = \mathrm{d}\lambda \frac{\partial F}{\partial A_k}, \tag{29}$$

where $\mathrm{d}\lambda$ is given by

$$\mathrm{d}\lambda = \frac{\mathrm{d}\varepsilon_{\mathrm{eq}}^{\mathrm{p}}}{\sqrt{\frac{2}{3}\frac{\partial F}{\partial \sigma_{ij}}\frac{\partial F}{\partial \sigma_{ij}}}}, \quad \text{where } \mathrm{d}\varepsilon_{\mathrm{eq}}^{\mathrm{p}} = \sqrt{\frac{2}{3}\mathrm{d}\varepsilon_{ij}^{\mathrm{p}}\mathrm{d}\varepsilon_{ij}^{\mathrm{p}}}. \tag{30}$$

Therefore, Eqs. (7), (8) and (21) give the dissipative energy density failure criterion as

$$\phi_1 = \int_0^{\varepsilon_{\mathrm{eq}}^{\mathrm{p}}} G\,\mathrm{d}\varepsilon_{\mathrm{eq}}^{\mathrm{p}} = \phi_{1c}, \tag{31}$$

in which,

$$G = \frac{\sigma_{ij}\frac{\partial F}{\partial \sigma_{ij}} + Y_{ij}\frac{\partial F}{\partial Y_{ij}} + A_k\frac{\partial F}{\partial A_k}}{\sqrt{\frac{2}{3}\frac{\partial F}{\partial \sigma_{ij}}\frac{\partial F}{\partial \sigma_{ij}}}}. \tag{32}$$

A large group of ductile failure criteria have been proposed in the form of Eq. (31). For example, when material plasticity and damage are considered separately, Zheng et al. (1994) obtained a similar expression of failure criterion, in which only damage term is kept. The plastic potential function F is expressed by

$$F(\sigma_{ij}, B) = \sigma_{\mathrm{e}} + g(B, \sigma_{\mathrm{m}}), \tag{33}$$

where B is the thermodynamics force associated with damage parameter D when isotropic damage is assumed. More examples are listed below:

(1) *Oyane (1972):*

$$G = \left(1 + \frac{1}{a_0}\frac{\sigma_{\mathrm{m}}}{\sigma_{\mathrm{e}}}\right)\left(\varepsilon_{\mathrm{eq}}^{\mathrm{p}}\right)^{c_0}, \tag{34}$$

where $\sigma_{\mathrm{m}} = 3\sigma_{\mathrm{H}}$, and a_0 and c_0 are material constants.

(2) *Zheng et al. (1994):*

$$G = \frac{1}{A}\exp\left(1.5\frac{\sigma_{\mathrm{m}}}{\sigma_{\mathrm{e}}}\right), \tag{35}$$

where A is material constant.

(3) *Chaouadi et al. (1994):*

$$G = \left[1 + \alpha\frac{\sigma_{\mathrm{m}}}{\sigma_{\mathrm{e}}}\exp\left(\frac{\sigma_{\mathrm{m}}}{2\sigma_{\mathrm{e}}}\right)\right]\sigma_{\mathrm{e}}, \tag{36}$$

where α is material constant.

(4) *Wang (1992a,b, 1994):*

$$G = F\left(\frac{\sigma_{\mathrm{H}}}{\sigma_{\mathrm{e}}}, \varepsilon_{\mathrm{eq}}^{\mathrm{p}}\right) = kf\left(\frac{\sigma_{\mathrm{H}}}{\sigma_{\mathrm{e}}}\right)\left(\varepsilon_{\mathrm{eq}}^{\mathrm{p}} - \varepsilon_0^{\mathrm{p}}\right)^{k-1}, \tag{37}$$

where k is a damage coefficient, $\varepsilon_0^{\mathrm{p}}$ is the void nucleation strain, and

$$f\left(\frac{\sigma_{\mathrm{H}}}{\sigma_{\mathrm{e}}}\right) = \frac{2}{3}(1+v) + 3(1-2v)\left(\frac{\sigma_{\mathrm{H}}}{\sigma_{\mathrm{e}}}\right)^2. \tag{38}$$

When $k = 1$ it is identical to Lemaitre and Chaboche (1990)'s results.

These failure criteria have been used in many applications when material damage plays an important role during structural failure. A large group of ductile failure criteria can be expressed within the framework of the general dissipative energy density failure criterion proposed in the present paper. However, it is necessary to identify the characteristics of a particular failure process in order to simplifying Eqs. (31) and (32).

4. Expression of plastic strain energy density failure criterion in dynamic structural responses

It has been observed that several structural elements, when subjected to sufficiently large transverse dynamic loads, may fail in one of three principal modes, i.e., (a) mode I: large inelastic deformations; (b) mode II: tearing (tensile failure); and (c) mode III: transverse shear failure, which were first observed by Menkes and Opat (1973) for beams.

A detailed examination of the experimental results reveals that modes II and III belong to a local tensile failure (Jones, 1989a) although with different response features. Therefore, the values of w_c^p for a plastic strain energy density failure criterion could be determined from a uniaxial tensile test with the proper consideration of strain rate and hydrostatic effects.

Rigid, perfectly plastic simplifications have been used successfully to predict the dynamic response of a wide range of structural elements (Jones, 1976, 1989b). It has advantages in understanding the deformation mechanisms and saves computer time when compared with numerical simulations. Shen and Jones (1992, 1993a,b) and Jones and Shen (1993) used the plastic strain energy density failure criterion to study various dynamical structural failures, which is described by

$$W^p = W_m + W_s + W_n = BHl_m w_c^p, \tag{39}$$

where W_m, W_s and W_n are the dissipated bending, shearing and membrane inelastic energies in a plastic hinge, w_c^p is the critical value of plastic strain energy density, B and H are beam width and thickness, and l_m is the plastic hinge length.

In this section, we will use the general plastic strain energy density failure criterion to obtain Eq. (39) for a beam made from rigid, perfectly plastic material. The method could be extended to plates and cylindrical shells.

Generalised stresses and strains could be introduced in order to simplify the analyses of beams and other structural elements. These generalised stresses and strains are defined with respect to the entire cross-section of a beam. However, a mode II failure, for example, is a local phenomenon, which does not occur simultaneously across the entire cross-section, but develops at a macroscopic point, as illustrated in Fig. 3(a). Such a macroscopic failure point is always located within a severely deformed zone where a mixture of bending, shearing and tensile deformations exists. This localised deformation zone is called a plastic hinge for a rigid, perfectly plastic beam. The plastic strain energy density at the expected failure point may be expressed as [4]

$$w^p = w_m + w_s + w_n, \tag{40}$$

where w_m, w_s and w_n are the plastic strain energy densities associated with the bending moment, shearing force and membrane force at the failure point A in Fig. 3(a), respectively. A rigid, perfectly plastic analysis does not provide any information on the structure of a plastic hinge. However, it is reasonable from an engineering viewpoint to assume that the stresses and deformations associated with bending, shearing and membrane hinges have the uniform distributions in the longitudinal direction within the hinge as shown in Fig. 3(b), (c) and (d). Therefore, the plastic strain energy density failure criterion expressed by Eq. (40) may be cast in the following form

$$W_m + \frac{l_m}{l_s} W_s + \frac{l_m}{l_n} W_n = BHl_m w_c^p, \tag{41}$$

because

[4] Here we assume that the deformations corresponding to the bending moment, transverse shearing force and the membrane force are separable. Therefore, the total deformations at a macroscopic point of a beam is the superimposition of the deformations at that point, which correspond to bending moment, transverse shearing force and membrane force defined in a beam cross-section.

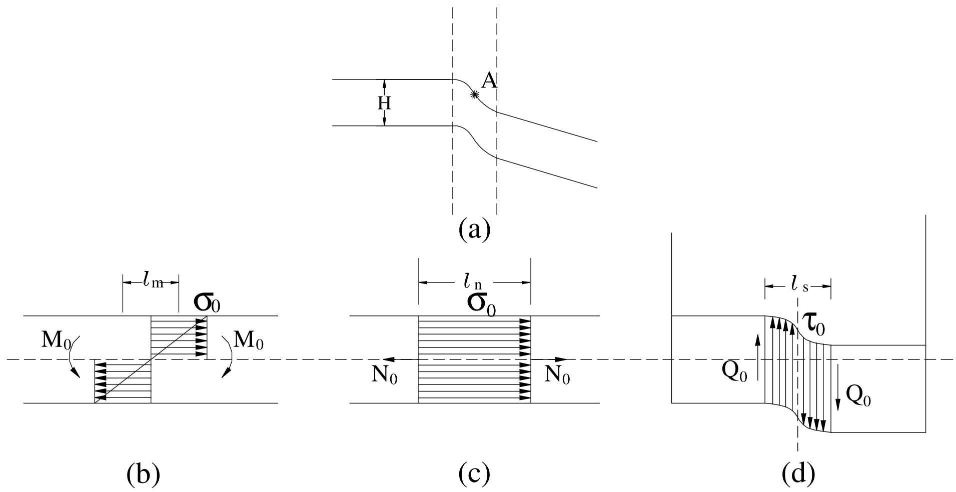

Fig. 3. (a) Macroscopic failure point in the deformed zone of a beam; (b) stress and strain distributions under bending; (c) stress and strain distributions corresponding to membrane force; (d) transverse shear deformation and stress distributions.

$$w_m = \frac{W_m}{BHl_m}, \quad w_s = \frac{W_s}{BHl_s} \quad \text{and} \quad w_n = \frac{W_n}{BHl_n}, \tag{42}$$

where B and H are the width and thickness of the beam, and W_m, W_s and W_n are the dissipated inelastic energies in the total length of the corresponding plastic hinges which have hinge lengths l_m, l_s and l_n, respectively. The value of w_c^p should be determined from a dynamic uniaxial tensile test. It should be noted that l_m is the average bending hinge length which is defined by the half-length of the bending deformation zone, as shown in Fig. 3. If it is assumed that $l_n = l_m$, then Eq. (41) becomes [5]

$$W^p = W_m + \gamma W_s + W_n = BHl_m w_c^p, \tag{43}$$

where $\gamma = l_m/l_s$. Eq. (43), in principle, may be extended to circular plates, cylindrical shells and other structural members in a similar way. Nonaka (1967) used slip line theory to study the behaviour of beams and obtained the average hinge length $l_m = H/2$ for a pure bending moment and $l_m = H$ at the beginning of a membrane state.

If $\gamma = 1$, Eq. (43) is simplified to Eq. (39), which has been used by Jones and Shen (1993). However, the assumption $\gamma = 1$ requires verification.

In the particular case of a rigid, perfectly plastic beam having a width B and a thickness H, the actual inelastic work absorbed at failure in a plastic hinge having an average length l_m across the beam thickness is

$$W_c^p = BHl_m w_c^p(\dot{\varepsilon}_m), \tag{44}$$

where $w_c^p(\dot{\varepsilon}_m)$ is determined from material test, $\dot{\varepsilon}_m$ is the mean strain rate.

The influence of material strain rate sensitivity on the critical value of the plastic strain energy density is unclear. w_c^p increases with strain rate according to the experimental data in Yu and Jones (1991) for an aluminium alloy. However, other source assumed that the value of w_c^p is insensitive to strain rate

[5] No generality is lost when it is assumed that $l_n = l_m$ because membrane force and the associated deformations are uniform along the beam. In this case, W_n is the plastic strain energy corresponding to the membrane force in an average bending hinge length.

(Jones, 1989c, 1993). Further systematic experiments are required at various strain rates to estimate the influence of strain rate effects on the value of w_c^p.

5. Relationship between failure criteria and fracture mechanics

It has been shown in Sections 2–4 that a strain energy failure criterion provides a general method for predicting the initiation of material failure. When a failure criterion is used to predict the possible initial failure sites in a solid continuum, it is unnecessary to consider the failure propagation interaction and its influence on stress and deformation fields. The subsequent occurrence of failure extension at the predicted failure site depends on the continuous energy supply through the given mechanical system whose boundary constrain depends inversely on the failure development. If the external energy supply is enough to develop a macroscopic crack in a certain direction, i.e., an energy balance when considering the energy required to generate new fracture surfaces is satisfied, the material failure will extend. Fracture mechanics, established by Griffith (1921), studies macroscopic crack extension through an energy balance method in a mechanical system, which may be described by

$$X\,\mathrm{d}u = \mathrm{d}W^e + \mathrm{d}W_r^e + \mathrm{d}\phi_1 + \mathrm{d}H + R\,\mathrm{d}A, \tag{45}$$

where X is the external load, u the loading point displacement, W^e the elastic strain energy of the mechanical system, W_r^e the residual elastic strain energy, ϕ_1 the inelastic dissipative energy defined by Eq. (8), H the kinetic energy of the mechanical system, A the macroscopic crack area and R the specific work of fracture (fracture toughness), as shown by Atkins (1988). It should be noted that R consists of both the surface energy and the dissipative energy in a boundary layer contiguous with the crack faces. Heat conduction is neglected in the analysis. Such an equation has been used in various cases and the contributions from different terms has been specified by Atkins (1988). For example, Griffith-based elastic fracture mechanics uses the solution of Eq. (45) when $\mathrm{d}W_r^e = \mathrm{d}\phi_1 = \mathrm{d}H = 0$. In rigid-plastic fracture mechanics, $\mathrm{d}W^e = \mathrm{d}W_r^e = \mathrm{d}H = 0$ and $\mathrm{d}\phi_1 = \mathrm{d}W^p$, where W^p is the plastic energy of the mechanical system.

When Eq. (45) is applied to a small zone adjacent to the crack, called a fracture process zone by Mai (1993), the input external work comes from the change of the stored elastic energy in the surrounding medium, i.e., the change of a specified elastic strain energy, $\mathrm{d}W_s^e$, which is released during fracture development. Within such an extensive inelastic deformation zone, the change of the kinetic energy in the mechanical system may be neglected for a stable fracture problem. In order to understand the actual physical meaning of the fracture toughness, R, it is necessary to separate the surface energy and the dissipative energy in the boundary layer of the crack faces, thus, Eq. (45) becomes

$$\mathrm{d}W_s^e = \mathrm{d}\Phi + \gamma\,\mathrm{d}A. \tag{46}$$

Free surface energy γ has clear definition here, i.e., the energy required to separate an atomic layer to create a unit free surface or fracture. The existence of the term $\mathrm{d}\Phi$ is based on the fact that there exists a thin layer adjacent to the fracture surface, in which inelastic dissipations are very extensive. This boundary layer is formed in the fracture process zone in front of the crack tip. The following definition of fracture toughness, R, includes contributions from both free surface and its boundary layers.

It is assumed that the distributions of releasable elastic strain energy and dissipative energy density are uniform in a characteristic volume of the fracture process zone. To create a free surface means to release the specified elastic strain energy in it, whose value should be equal to its critical value for initiating a material failure, as suggested by Eq. (9). Thus,

$$\mathrm{d}W_s^e = w_{sc}^e\,\mathrm{d}V \quad \text{and} \quad \mathrm{d}\Phi = \phi\,\mathrm{d}V, \tag{47}$$

where w_{sc}^{e} is the critical value of a specified elastic strain energy density defined in Eq. (9) and ϕ is the dissipative energy density in fracture process zone, $\mathrm{d}V$ is a characteristic volume of the fracture process zone. Such an understanding establishes a relationship between the free surface energy and the critical specified elastic strain energy density. Eqs. (46) and (47) give

$$\phi\frac{\mathrm{d}V}{\mathrm{d}A}+\gamma=w_{sc}^{e}\frac{\mathrm{d}V}{\mathrm{d}A} \tag{48}$$

and thus, the fracture toughness, R, defined by the energy consumption to create a new unit area within the fracture process zone, is

$$R=\frac{\mathrm{d}W_{s}^{e}}{\mathrm{d}A}=\lambda\phi+\gamma=\lambda w_{sc}^{e}, \tag{49}$$

where $\lambda=\Delta V/\Delta A$ is introduced here as the characteristic thickness of the boundary layer of a fracture surface, which is an inherent characteristic length of a material failure. The actual value of λ may depend on the state and internal variables of the deformed material and the crack modes, which makes the problem more complicated, and further study is necessary.

Because macroscopic material failure initiates from an infinitesimal element (or a macroscopic point) the strain energy density is suitable for establishing a macroscopic failure criterion as discussed in Sections 2–4. During the crack propagation process represented by a new generated surface, fracture toughness, R, would be more proper than a strain energy density to describe crack propagation in an energy balance equation (45) and it is also easier to be determined from experimental tests in commonly observed fracture modes. Therefore, in material failure analyses, a strain energy density failure criterion is used to describe the material failure initiation. In failure propagation analyses (elastic–plastic fracture mechanics), the fracture toughness with an energy balance method is adopted. However, in case when R depends on the preceding inelastic deformations as w_{sc}^{e} does, fracture mechanics still faces difficulties to study crack propagation. Fortunately, in many cases, macroscopic cracks propagate in three basic modes or their combinations, where the preceding inelastic dissipations at the tip of these cracks are well defined. If ϕ_{I}, ϕ_{II} and ϕ_{III} are the inelastic dissipative energy densities at the crack tip for the three basic crack modes, we have $R_{\mathrm{I}}=R(\phi_{\mathrm{I}})$, $R_{\mathrm{II}}=R(\phi_{\mathrm{II}})$ and $R_{\mathrm{III}}=R(\phi_{\mathrm{III}})$ which may be thought approximately as constants during the crack propagation process. The concept of essential fracture work proposed by Mai (1993) is a good example. However, in general, stain rate, temperature and the change of ϕ may influence their value. Thus, although R is treated as a material constant in fracture mechanics, its dependence on strain rate, temperature and even specimen thickness are observed and emphasised in many investigations because these factors may influence R either directly or through their influence on ϕ in the crack boundary layers. Furthermore, the understanding of ϕ distributions around the crack tip (or in the crack process zone) is necessary in fracture mechanics for calculating the inelastic dissipative energy in an energy balance equation. A combination between the analyses of material failure initiation and crack propagation is necessary to give a complete solution of material failure process.

Historically, failure analysis and fracture mechanics are studied separately. Principally, failure analysis could be used to predict failure extensions in any circumstances, and thus, it is more general than fracture mechanics. But, the actual operation involves great difficulties in both theoretical and numerical processes because the boundary conditions, which are required for checking the failure criterion, depends on the failure extension that is in turn the result of a failure criterion. Due to this strong nonlinearity, fracture mechanics based on global energy balance is adopted to solve some well-defined crack propagation problems, which have been established as a significant subject in mechanics and engineering fields.

6. Conclusions

A method based on continuum thermodynamics is presented in the present paper for using energy density concept to define a failure criterion. Material failure in solid material is triggered by internal interactions between material particles, which is represented macroscopically by a specified elastic strain energy density. However, the critical value of such a specified elastic strain energy density is highly depended on the previous irreversible dissipation. Generally, material brittle failure is mainly controlled by material internal interactions or the specified elastic strain energy density while material ductile failure is determined mainly by plastic and damage dissipation. There are other cases where both of them are important.

Many previously successful failure criteria could be embedded in the current analysis. However, due to the extremely complex failure phenomena, considerable material tests are required to determine the actual function and parameters in a particular failure criterion. Furthermore, the assumptions and conclusions need to be verified by well-designed material tests, which is the principal means to judge the suitability of a practical failure criterion.

Acknowledgements

The author acknowledges Professor Norman Jones for his comments on the paper and acknowledges Mr. H. Meng for preparing all figures.

References

Alves, M., 1996. Damage mechanics applied to structural impact. Ph.D. Thesis, University of Liverpool.

Atkins, A.G., 1988. Scaling in combined plastic flow and fracture. Int. J. Mech. Sci. 30, 173–191.

Bazant, Z.P., Chen, E.P., 1997. Scaling of structural failure. Appl. Mech. Rev. 50 (10), 593–627.

Becker, R., 1987. The effect of porosity distribution on ductile fracture. J. Mech. Phys. Solids 35, 577–599.

Benallal, A., Comi, C., Lemaitre, J., 1992. Critical damage states at crack initiation. In: Ju, J.W., Valanis, K.C. (Eds.), Damage Mechanics and Localization, AMD-Vol. 142/MD-Vol. 34, ASME.

Bridgman, P.W., 1952. Studies in Large Plastic Flow and Fracture. McGraw-Hill, New York.

Chaouadi, R., De Meester, P., Vandermeulen, W., 1994. Damage work as ductile fracture criterion. Int. J. Fract. 66, 155–164.

Chrysochoos, A., Belmahjoub, F., 1992. Thermographic analysis of thermomechanical coupling. Arch. Mech. 44, 55–68.

Clift, S.E., Hartley, P., Sturgess, C.E.N., Rowe, G.W., 1987. Further studies on fracture initiation in plane-strain forging. In: Tooth, A.S., Spence, J. (Eds.), Solid Mech. 2. Elsevier, Amsterdam.

Clift, S.E., Hartley, P., Sturgess, C.E.N., Rowe, G.W., 1990. Fracture prediction in plastic deformation process. Int. J. Mech. Sci. 32, 1–17.

Cockcroft, M.G., Latham, D.J., 1968. Ductility and the workability of metals. J. Inst. Metals 96, 339.

Cottrell, A.H., 1964. The Mechanical Properties of Matter. Wiley, London.

Dodd, B., Bai, Y.L., 1987. Ductile Fracture and Ductility – with Application to Metalworking. Academic Press, London.

Dorn, J.E., 1948. The effect of stress state on the fracture strength of metals, Fracture of Metals. ASM, pp. 32–50.

Freudenthal, A.M., 1950. The Inelastic Behaviour of Solids. Wiley, New York.

Gillemot, L.F., 1976. Criterion of crack initiation and spreading. Engng. Fract. Mech. 8, 239–253.

Gordon, J.E., 1976. The New Science of Strong materials, second ed., Penguin Books, Baltimore.

Griffith, A.A., 1921. The phenomena of rupture and flow in solid. Phil. Trans. R. Soc. A 221, 163–197.

Gurson, A.L., 1977. Continuum theory of ductile rupture by void nucleation and growth: part I – Yield criteria and flow rules for porous ductile media. J. Engng. Mater. Tech. 99, 2–15.

Hancock, J.W., Mackenzie, A.C., 1976. On the mechanism of ductile failure in high strength steels subjected to multi-axial stress-states. J. Mech. Phys. Solids 24, 147–169.

Hancock, J.W., Brown, D.K., 1983. On the role of strain and stress state in ductile failure. J. Mech. Phys. Solids 31, 1–24.

Hill, R., 1948. A variational principle of maximum plastic work in classical plasticity. Quart. J. Mech. Appl. Math. 1, 18–28.

Jones, N., 1976. Plastic failure of ductile beams loaded dynamically. Trans. ASME, J. Engng. Ind. 98, 131–136.
Jones, N., 1989a. On the dynamic inelastic failure of beams. In: Wierzbicki, T., Jones, N., Structural Failure. Wiley, New York, pp. 133–159.
Jones, N., 1989b. Structural Impact. Cambridge University Press, Cambridge.
Jones, N., 1989c. Some comments on the modelling of material properties for dynamic structural plasticity. In: Harding, J. (Ed.), Mechanical Properties of Materials at High rates of Strain., Inst. Phys. Conf. Series no. 102, Bristol, pp. 435–445.
Jones, N., 1993. Material properties for structural impact problems. In: Rama Rao, P. (Ed.), Advances in Materials and Their Applications. Wiley Eastern Limited, New Delhi, pp. 151–163.
Jones, N., 1997. Dynamic inelastic failure of structures. Trans. JSME 63, 2485–2495.
Jones, N., Shen, W.Q., 1993. Criteria for the inelastic rupture of ductile metal beams subjected to large dynamic loadings. In: Jones, N., Wierzbicki, T. (Eds.), Structural Crashworthiness and Failure. Elsevier Appl. Sci., London, pp. 96–130.
Lee, E.H., 1969. Elastic-plastic deformation at finite strains, Trans. ASME, J. Appl. Mech. 36, 1–6.
Lee, E.H., Liu, D.T., 1967. Finite-strain elastic-plastic theory particularly for plane-wave analysis. Trans. ASME, J. Appl. Mech. 34, 19–27.
Lemaitre, J., Chaboche, J.L., 1990. Mechanics of Solid Materials. Cambridge University Press, Cambridge.
Li, Q.M., 1999. Dissipative flow model based on dissipative surface and irreversible thermodynamics. Arch. Appl. Mech. 69, 379–392.
Li, Q.M., 2000. Energy correlation between a damaged macroscopic continuum and its subscale. Int. J. Solids Struct. 37, 4539–4556.
Mai, Y.W., 1993. On the plane-stress essential fracture work in plastic failure of ductile material. Int. J. Mech. Sci. 35, 995–1005.
Malvern, L.E., 1969. Introduction to the Mechanics of a Continuous Medium. Prentice-Hall, NJ.
Matic, P., Kirby III, G.C., Jolles, M.I., 1988. The relation of tensile specimen size and geometry effects to unique constitutive parameters for ductile materials. Proc. R. Soc. Lond. A 417, 309–333.
Menkes, S.B., Opat, H.J., 1973. Broken beams. Exp. Mech. 13, 480–486.
Nadai, A., 1950. Theory of Flow Fracture of Solids, vol. 1. McGraw-Hill, London.
Nonaka, T., 1967. Some interaction effects in a problem of plastic beam dynamics. ASME J. Appl. Mech. Part 1–3 34, 623–643.
Otsuka, A., Tohgo, K., Okamoto, Y., 1987. Relationship between ductile crack initiation and void volume fraction. Nucl. Engng. Des. 105, 121–129.
Oyane, M., 1972. Criteria of ductile fracture strain. Bull. JSME 15, 1507–1513.
Polakowski, N.H., Ripling, E.J., 1966. Strength and Structure of Engineering Materials. Prentice-Hall, Englewood Cliffs, NJ.
Shen, W.Q., Jones, N., 1992. A failure criterion for beams under impulsive loading. Int. J. Impact Engng. 12, 101–121 and 329.
Shen, W.Q., Jones, N., 1993a. Dynamic plastic response and failure of a clamped beam struck transversely by a mass. Int. J. Solid Struct. 30, 1631–1648.
Shen, W.Q., Jones, N., 1993b. Dynamic response and failure of fully clamped circular plates under impulsive loading. Int. J. Impact Engng. 13, 259–278.
Shi, Y.W., Barnby, J.T., Nadkarni, A.S., 1991. Void growth at ductile crack initiation of a structural steel. Engng. Fract. Mech. 39, 37–44.
Tvergaard, V., 1990. Material failure by void coalescence. In: Hutchinson, J.W., Wu, T.Y. (Eds.), Advances in Applied Mechanics. Academic Press, New York, pp. 27, 83–151.
Vinh, T., Khalil, T., 1984. Adiabatic and viscoplastic properties of some polymers at high strain and high strain rate. Institute of Physics Conference Series n70, pp. 39–46.
Wang, T.J., 1992a. Unified CDM model and local criterion for ductile fracture – I: Unified CDM model for ductile fracture. Engng. Fract. Mech. 42, 177–183.
Wang, T.J., 1992b. Unified CDM model and local criterion for ductile fracture – II: Ductile fracture local criterion based on the CDM model. Engng. Fract. Mech. 42, 185–193.
Wang, T.J., 1994. Further investigation of a new continuum damage mechanics criterion for ductile fracture: experimental verification and applications. Engng. Fract. Mech. 48, 217–230.
Woo, C.W., Li, D.L., 1993. A universal physically consistent definition of material damage. Int. J. Solids Struct. 30, 2097–2108.
Yu, J.L., Jones, N., 1991. Further experimental investigations on the failure of clamped beams under impact loads. Int. J. Solids Struct. 27, 1113–1137.
Zheng, M., Deng, Z., Jin, Z., 1994. On the ductile damage theory. Int. J. Fract. 67, R87–R92.